AUGUST 2008 ANALYSIS QUALIFYING EXAM

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1. Problem 1

(a). Let $x_n \to x$ be a sequence of points in $A + B$. We want to show that $x \in A + B$; note that $x_n = a_n + b_n$ for $b_n \in A$, $a_n \in A$.

By compactness, we may choose a convergent subsequence a_{n_j} such that $a_{n_j} \to a \in A$ as $j \to \infty$. Consider then the subsequence $b_{n_j} = x_{n_j} - a_{n_j}$; as $j \to \infty$, this converges to $x - a$, and since B is closed, $x - a \in B$.

We then note that $x = (x - a) + a$ is an element of the Minkowski sum $A + B$.

(b). Set $A = \mathbb{Z}$, $B = \sqrt{2}\mathbb{Z}$. Then, $\overline{A + B} = \mathbb{R}$, but obviously $A + B \neq \mathbb{R}$, so $A + B$ is not closed.

2. Problem 2

(1) $\implies$ (2): SUppose T is continuous at a. Then, for all $\epsilon > 0$, there exists δ such that

$$||a - y|| < \delta \implies ||Ta - Ty|| < \epsilon$$

Suppose then that $||x - y|| < \delta$. Then, as

$$||x - y|| = ||a - (y - x + a)|| < \delta$$

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we have

$$||Ta - T(y - x + a)|| = ||Tx - Ty|| < \epsilon$$

so that T is continuous.

(2) $\implies$ (3): By continuity at 0, there exists $\delta > 0$ such that for all $x \in X$ with $||x|| < \delta$, $||Tx|| \leqslant 1$. For arbitrary x, we see

$$||T(x)|| = \Big|\Big|\frac{||x||}{\delta} \cdot T\Big(\frac{\delta x}{||x||}\Big)\Big|\Big| \leqslant \frac{1}{\delta}||x||$$

So we may take $M := \frac{1}{\delta}$.

(3) $\implies$ (1): Let $\epsilon > 0$. If $||Tx|| \leqslant M||x||$, choose $\delta := \frac{\epsilon}{M+1}$. Then, whenever $||x|| < \delta$,

$$||Tx|| < \frac{M\epsilon}{M+1} < \epsilon$$

so that T is continuous at $a = 0$, completing the proof.

3. Problem 3

Replacing f and g by $f/||f||_p$ and $g/||g||_q$ respectively, we may assume by homogeneity that $||f||_p = ||g||_q = 1$ (note that if either norm vanishes the result is trivial).

By Young's inequality,

$$\begin{aligned} ||fg||_1 &= \int_E |fg| d\mu \\ &\leqslant \int_E \frac{|f|^p}{p} + \frac{|g|^q}{q} d\mu \\ &= \frac{||f||_p^p}{p} + \frac{||g||_q^q}{q} \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

(b). If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, we can translate this into the situation of part (a) by simply multiplying by r; then $\frac{1}{p/r} + \frac{1}{q/r}$, and

$$\begin{aligned} ||fg||_r^r &= \int_E |f^r g^r| d\mu \\ &\leqslant \Big(\int_E |f|^p \Big)^{r/p} \Big(\int_E |g|^q d\mu \Big)^{r/q} \end{aligned}$$

Taking rth roots in the above, we see

$$||fg||_r \leqslant ||f||_p ||g||_q$$

4. Problem 4

(a). Let $\epsilon > 0$. We first prove the statement for simple functions. Set $s := \sum_{k=1}^N a_k \chi_{E_k}$, and let $M := \max_k \{|a_k|\}$. Choose A with $\mu(A) < \frac{\epsilon}{NM}$.

Then,

$$\begin{aligned} \int_A s d\mu &= \sum_{k=1}^N a_k \mu(E_k \cap A) \\ &< \frac{\epsilon}{NM} \sum_{k=1}^N |a_k| \\ &\leqslant \epsilon \end{aligned}$$

Now, for the general case, assume without loss of generality that $f \geqslant 0$. Let $\epsilon > 0$; by definition of Lebesgue integral we may choose $s \leqslant f$ a simple function such that

$$\int_{\mathbb{R}} f - s d\mu < \frac{\epsilon}{2}$$

Since s is simple, we may choose δ such that $\mu(A) < \delta$ implies $\int_A s d\mu < \epsilon/2$ by the above.

Then, for $\mu(A) < \delta$,

$$\int_A f d\mu = \int_A f - s d\mu + \int_A s d\mu$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Which gives the result.

(b). Let $\epsilon > 0$. By definition of supremum, there exists $N \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \int_E |f_n| d\mu - \int_E |f_N| d\mu < \epsilon/2$$

By part (a), since each f_n is integrable, there exists δ such that $\mu(A) < \delta$ implies $\int_A |f_N| d\mu < \epsilon/2$.

Then, let $\mu(A) < \delta$. We see:

$$\sup_{n\in\mathbb{N}} \int_E |f_n| d\mu < \frac{\epsilon}{2} + \int_A |f_n| d\mu$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

5. Problem 5

(a). Note first that since $|f_n| \leqslant g$ for all n, letting $n \to \infty$ gives $|f| \leqslant |g|$ as well. By Fatou's lemma, we see

$$0 \leqslant \int_X 2^p - \lim_{n\to\infty} |f_n - f|^p d\mu$$
$$\leqslant \liminf_{n\to\infty} \Big(\int_X 2^p g - \int_X |f_n - f| d\mu \Big)$$
$$= \int_X 2^p g - \limsup_{n\to\infty} \int_X |f_n - f|^p d\mu$$
$$\implies \limsup_{n\to\infty} \int_X |f_n - f|^p d\mu \leqslant 0$$

So,

$$||f_n - f||_p \to 0$$

and, by the triangle inequality we see $||f_n||_p \to ||f||_p$, as desired.

(b). For every p, set

$$f_n := \begin{cases} \left(\frac{-n^2}{2}x + n\right)^{1/p}, & x \in [0, 1/n] \\ 0, & x \in [1/n, 1] \end{cases}$$

Then, $f_n \to 0$ almost everywhere. However, it is easy to see that $||f_n||_p = 1$ for all n and p, which certainly does not tend to 0.

6. Problem 6

(a). Note that

$$\begin{aligned} |\widetilde{f}(\xi)| &\leqslant \int_{\mathbb{R}} |f(x)| dx \\ &= ||f||_1 < \infty \end{aligned}$$

So that $\widetilde{f}$ exists an is bounded. For continuity, note that

$$||\widetilde{f}(\xi + h) - \widetilde{f}(\xi)| \leqslant 2||f||_1$$

So that by Lebesgue's dominated convergence theorem,

$$\begin{aligned} \lim_{h \to 0} \int_{\mathbb{R}} e^{ix\xi}\big(e^{ixh} - 1\big) d\mu &= \int_{\mathbb{R}} \lim_{h \to 0} e^{ix\xi}\big(e^{ixh} - 1\big) d\mu \\ &= 0 \end{aligned}$$

So that $\widetilde{f}$ is continuous.

(b). Note that by part (a) we have that

$$||\widetilde{f}||_\infty \leqslant ||f||_1$$

whence by Hölder's inequality,

$$\begin{aligned} \int_{\mathbb{R}} \widetilde{f}(\xi) g(\xi) d\xi &\leqslant ||\widetilde{f}||_\infty ||g||_1 \\ &\leqslant ||f||_1 ||g||_1 \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}} f(\xi) \widetilde{g}(\xi) d\xi &\leqslant ||f||_1 ||\widetilde{g}||_\infty \\ &\leqslant ||f||_1 ||g||_1 \end{aligned}$$

So that both integrals exist and are bounded. Now,

$$\begin{aligned}\int_{\mathbb{R}} \widetilde{f}(\xi)g(\xi)d\xi &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{ix\xi}dxg(\xi)d\xi \\ &= \int_{\mathbb{R}} f(x)\int_{\mathbb{R}} e^{ix\xi}g(\xi)d\xi \quad \text{(Fubini-Tonelli)} \\ &= \int_{\mathbb{R}} f(x)\widetilde{g}(x)dx\end{aligned}$$

Which was to be proved.

7. Problem 7

Assume $|f(z)| \leqslant M$. By holomorphicity, we have a power series expansion

$$f(z) = \sum_{n\geqslant 0} a_n z^n$$

where

$$a_n = \frac{1}{2\pi i}\int_{B_r(0)} \frac{f(z)}{z^{n+1}}dz$$

Consider now for $n \geqslant 1$,

$$\begin{aligned}|a_n| &\leqslant \frac{1}{2\pi}\int_{B_r(0)} \frac{|f(z)|}{|z|^{n+1}}dz \\ &= \frac{1}{2\pi r^{n+1}}\int_{B_r(0)} |f(z)|dz \\ &\leqslant \frac{1}{2\pi r^{n+1}} \cdot M \cdot 2\pi r \\ &= \frac{M}{r^n}\end{aligned}$$

As f is entire, we may take $r \to \infty$ to find that $|a_n| = 0$ for all $n \geqslant 1$; that is, $f \equiv a_0$, so that f is constant.

8. Problem 8

(a). By continuity of f, give $\epsilon > 0$ there exists δ such that whenever $|\theta - t| < 2\delta$, $|f(t) - f(\theta)| < \epsilon/3$. Then,

$$\begin{aligned}\int_{-\pi}^{\pi} f(\theta - t)P_r(t)dt &= \int_{-\pi}^{\theta-\delta} f(\theta - t)P_r(t)dt \\ &+ \int_{\theta-\delta}^{\theta+\delta} f(\theta - t)P_r(t)dt \\ &+ \int_{\theta+\delta}^{\pi} f(\theta - t)P_r(t)dt \\ &:= I_1 + I_2 + I_3\end{aligned}$$

Then, by our selection of δ,

$$|I_2| \leqslant \frac{\epsilon}{3} \cdot \int_{-\pi}^{\pi} P_r(t)dt = \epsilon/3$$

Also, as $[-\pi, \pi]$ is compact and f is continuous, $|f| \leqslant M$, so that

$$\begin{aligned}|I_1| &\leqslant M \int_{-\pi}^{\theta-\delta} P_r(t)dt \\ &\leqslant 2\pi M \cdot \frac{(1 - r^2)}{(1 - r\cos(\delta))^2} \\ &< \epsilon/3\end{aligned}$$

whenever $|1 - r| < \frac{(1-\cos(\delta))^2}{12\pi M} \cdot \epsilon$. In an identical manner, we also see

$$|I_3| < \epsilon/3 \text{ for } \frac{(1 - \cos(\delta))^2}{12\pi M} \cdot \epsilon$$

Then,

$$\int_{-\pi}^{\pi} |f(\theta - t) - f(\theta)||P_r(t)|dt < |I_1| + |I_2| + |I_3| < \epsilon$$

Note of course that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)P_r(t)dt = f(\theta)$$

In which case,

$$\lim_{r \to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - t)P_r(t)dt = f(\theta)$$

(b). Yes, this is uniform. Let $\epsilon > 0$; we may find θ' such that

$$\sup_{0 \leqslant \theta \leqslant 2\pi} |(f * P_r)(\theta) - f(\theta)| < \epsilon/2 + |(f * P_r)(\theta') - f(\theta')|$$

Now, take the limit as $r \to 1^-$ in the above to find

$$\lim_{r \to 1^-} |(f * P_r)(\theta) - f(\theta)| < \epsilon/2 < \epsilon$$

Whence the result.

(c). The solution u may be found as $u(r,\theta) := (f * P_r)(\theta)$. It remains only to see that this is harmonic; the other properties follow easily from the above two parts. Recall that in polar coordinates the Laplacian is

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Then,

$$\begin{aligned}
r\frac{\partial}{\partial r}\Big(\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\Big) &= \frac{-2r^2}{1-2r\cos(\theta-t)+r^2} \\
&\quad + \frac{(1-r^2)(2r^2-2r\cos(\theta-t))}{(1-22\cos(\theta-t)+r^2)^2} \\
\implies \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big)\Big(\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\Big) &= \frac{-4}{1-2r\cos(\theta-t)+r^2} \\
&\quad + \frac{2r(-2\cos(\theta-t)+2r)}{(1-2r\cos(\theta-t)+r^2)^2} \\
&\quad + \frac{-2r(-2\cos(\theta-t)+2r)}{(1-2r\cos(\theta-t)+r^2)^2} \\
&\quad + \frac{(1-r^2)(4r-2\cos(\theta-t)}{r(1-2r\cos(\theta-t)+r^2)^2} \\
&\quad - \frac{2(1-r^2)(2r-2\cos(\theta-t))^2}{(1-2r\cos(\theta-t)+r^2)^3} \\
&= \frac{2(2\cos^2(\theta-t)r+\cos(\theta-t)r^2+\cos(\theta-t)-4r)(1-r^2)}{2(1-2r\cos(\theta-t)+r^2)^3}
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial \theta}\Big(\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\Big) = \frac{-2(1-r^2)r\sin(\theta-t)}{(1-2r\cos(\theta-t)+r^2)^2}$$

so that

$$\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big(\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\Big)$$
$$=\frac{-2(2\cos^2(\theta-t)r+\cos(\theta-t)r^2+\cos(\theta-t)-4r)(1-r^2)}{2(1-2r\cos(\theta-t)+r^2)^3}$$

And, adding those together clearly gives 0. Then, differentiation under the integral sign, we then deduce that $u(r,\theta)$ is harmonic as desired, as we have solved the Dirichlet problem.